

Quantum Dissension: Generalizing Quantum Discord for Three-Qubit States

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We introduce the notion of quantum dissension for a three-qubit system as a measure of quantum correlations. We use three classically equivalent expressions of three-variable mutual information. Their differences are zero classically but not so in quantum domain. It generalizes the notion of quantum discord to a multipartite system. There can be multiple definitions of the dissension depending on the nature of projective measurements done on the subsystems. As an illustration, we explore the consequences of these multiple definitions and compare them for three-qubit pure and mixed GHZ and W states. We find that unlike discord, dissension can be negative. This is because measurement on a subsystem may enhance the correlations in the rest of the system. Furthermore, when we consider a bipartite split of the system, the dissension reduces to discord. This approach can pave a way to generalize the notion of quantum correlations in the multiparticle setting.

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1. INTRODUCTION

Quantum entanglement plays an important role in the quantum communication protocols like teleportation [1, 2], superdense coding [3], remote state preparation [4], cryptography [5] and many more. However, the precise role of entanglement in quantum information processing still remains an open question. It is not clear, whether all the information processing tasks can be done more efficiently with a quantum system that requires entanglement as a resource. In addition, precise nature of the quantum correlations is not well understood for two-qubit mixed states and multipartite states. It has been suggested that the quantum correlations go beyond the simple idea of entanglement. The idea of quantum discord [6–8] is to quantify all types of quantum correlations including entanglement. It must be emphasized here that discord actually supplements the measure of entanglement that can be defined on the system of interest. Other measures of quantum correlations that have been proposed in the literature similar to discord are quantum deficit [9–11], quantumness of correlations [12] and quantum dissonance [13].

The idea of discord [6] uses the generalization of two-variable mutual information to quantum domain. The difference of two classically equivalent expressions when generalized to quantum setting gives the discord. It was shown that the discord reduces to von Neumann entropy for a pure bipartite state. Furthermore, it was found to be nonzero for some of the separable states. A number of authors have computed the discord in diverse situations [14]. In a different approach, entropic methods have been proposed to understand the separability and correlations of a composite state [9]. In particular, ‘quantum deficit’ was introduced as a measure of quantumness over classical correlations. Quantum deficit is the difference of the von Neumann entropy of the system and that of the decohered state. In another approach, Horodecki *et al* [10] have investigated the relation between *local* and *nonlocal* information by investigating a situation where parties sharing a multipartite state distill local

information. The amount of information that is lost due to the use of classical communication channel is called *deficit*. It was shown that the upper bound for the deficit is given by the relative entropy distance to so-called pseudo classically correlated states. On the other hand the lower bound is the relative entropy of entanglement. It was argued that it was the basic reason for any entangled state to be viewed as informationally nonlocal. In another piece of work [11], a simple information-theoretical measure of the one-way distillable local purity was introduced. Interestingly, the author showed that his proposed characterization is closely related to a previously known operational measure of classical correlations, the one-way distillable common randomness.

Recently, another measure has been introduced which is called quantumness of correlation [12]. It has been defined for bipartite states by incorporating a specific measurement scheme. It was shown that if one uses the optimal generalized measurement on one of the subsystem it reduces the overall state in its closest separable form. It was seen that this measure gives a non zero value for all bipartite entangled states and zero for separable states. Not only that, it also serves as an upper bound to the relative entropy of entanglement. In Ref [13], authors have given a unified view of the correlations in a given quantum state by classifying it into entanglement, dissonance, and classical correlations. They used the concept of relative entropy as a distance measure of correlations. It has been claimed that their methods completely fit into multipartite systems of arbitrary dimensions. In addition they showed that dissonance attains a non zero value in the case of pure multipartite states. It has also been suggested [15] that one single real number may not be able to capture the multifaceted nature of the correlations in the multipartite situation. One may need a vector quantity (a set of numbers) to characterize the correlations. In addition, the notion of maximally entangled state may depend on the task that is to be carried out.

In this paper, we generalize the notion of quantum discord from bipartite to tripartite systems. Our approach is based on three-variable mutual information and its generalization to

quantum setting. Classically, three-variable mutual information characterizes the common information contained in three classical distributions. Therefore, it appears natural to generalize it to quantum domain. We introduce three expressions of the three-variable mutual information, all of which are same classically but differ when conditional entropies are generalized to quantum level. For a tripartite system, one can make measurement on one-particle, or on more than one-particle to probe the different aspects of the quantum correlations. This would lead to multiple quantities that can characterize the correlations. We call these physical quantities ‘‘Quantum Dissension.’’ In the case of a tripartite state, we shall have two quantities that will characterize the correlations together. One can think of the dissension as the information contained in the multiparticle quantum state that cannot be extracted by either one-particle or two-particle measurements. We note, as discussed below, in the case of one-particle measurement, the dissension can be negative because a measurement on a subsystem can enhance the correlations in the rest of the system. Interestingly, in the case of two-particle measurements, the dissension is just the discord of a bipartite split of the system. We illustrate the idea of dissension with the help of specific three-qubit pure and mixed states. The pure states have genuine multipartite correlations. These states have non-zero dissension with respect to both one and two-particle measurements. Interestingly, in the case of mixed GHZ and W states, we have situations analogous to the two-qubit Werner state. It is known that Werner state has non-zero discord even for those values of classical mixing parameter for which it is not entangled. Similarly, we find that the mixed GHZ and W states have non-zero dissension for all values of classical mixing parameter. Therefore, dissension may be characterizing the quantumness in multiparticle system that goes beyond entanglement. We also provide an example where dissension due to two-particle measurement is zero, but it is non-zero when we make only one-particle measurement.

The organization of our paper is as follows. In section 2, we introduce the notion of three-variable mutual information and then define the quantum dissensions for one-particle and two-particle measurements. In section 3, we calculate the dissension for pure three-qubit states like GHZ and W states. In section 4, we illustrate the dissension for generalized Werner kind of three-qubit mixed states. We also discuss the case of a biseparable state. Finally, in section 5, we conclude and mention future directions of explorations.

2. QUANTUM DISSENTION

In classical information theory [16], one can quantify the relationship between two random variables X and Y by a quantity called mutual information $I(X : Y) = H(X) - H(X|Y)$ where $H(X)$, $H(X|Y)$ are the entropy of X and the conditional entropy of X given that Y has already occurred. Mutual information actually gives the measure of the reduction in the uncertainty about one random variable be-

cause of the occurrence of the other random variable. Since $H(X|Y) = H(X, Y) - H(Y)$, so there is another alternative expression for the mutual information $J(X : Y) = H(X) + H(Y) - H(X, Y)$. Classically, these two expressions of the mutual information are identical. We can generalize these expressions to quantum domain by substituting random variables X, Y by the density matrices ρ_X, ρ_Y and the Shannon entropies $H(X), H(Y)$ by the Von Neumann entropies (e.g: $H(X) = H(\rho_X) = -\text{Tr}[\rho_X \log(\rho_X)]$). With these substitutions, one can define J in the quantum case. But to define I , one needs to specify the conditional entropy $H(X|Y)$. To specify this conditional entropy, the measurement of Y is defined by a set of one dimensional projectors $\{\pi_i^Y\}$ [6]. Here, the subscript i refers to the outcome of this measurement. This gives us the quantum analogue of $I(X : Y)$ as

$$I(X : Y) = H(X) - H(X|\{\pi_i^Y\}), \quad (1)$$

where $H(X|\{\pi_i^Y\}) = \sum_j p_j H(\rho_{X|\pi_i^Y})$, $\rho_{X|\pi_i^Y} = \frac{\pi_i^Y \rho_{XY} \pi_i^Y}{\text{Tr}(\pi_i^Y \rho_{XY})}$ and p_i is the probability of obtaining the i th outcome. It is clearly evident that these two expressions for $I(X : Y)$ and $J(X : Y)$ are not identical in quantum theory. The quantum discord is the difference [6–8]

$$D(X : Y) = J - I = H(Y) - H(X, Y) + H(X|\{\pi_i^Y\}). \quad (2)$$

This is to be minimized over all sets of one dimensional projectors $\{\pi_i^Y\}$.

It is possible to generalize the discord to multipartite systems by considering multiparticle-measurement. In this paper, we introduce the notion of Quantum Dissension in the context of three-qubits. In the case of three qubits, there can be two types of projective measurements. These are one-particle projective measurements and two-particle projective measurements. These measurements can be performed on different subsystems. This would lead to multiple definitions of the quantum dissension. We focus on two scenarios. In one case, we consider all possible one-particle projective measurements. In the second scenario, we make all possible two-particle measurements.

To obtain the definition of the dissension for these two scenarios, we consider three-variable classical mutual information [16]. It is defined as

$$I(X : Y : Z) = I(X : Y) - I(X, Y|Z). \quad (3)$$

Here, $I(X, Y|Z)$ is the conditional mutual information

$$I(X, Y|Z) = H(X|Z) + H(Y|Z) - H(X, Y|Z). \quad (4)$$

Both $I(X : Y)$ and $I(X, Y|Z)$ are non-negative. However, there may exist a situation, when the conditional mutual information is greater than the mutual information. It happens when knowing the variable Z enhances the correlation between X and Y . In such a case, the three-variable mutual information is negative. One well known example where it occurs is modulo 2 addition of two binary random variables.

This is XOR gate. Suppose we add X and Y variables and get the variable Z . Let the variables X and Y be independent, then $I(X : Y)$ is zero. However, once we know the value of Z , knowing the value of Y determines the value of X uniquely. So the knowledge of Z , enhances the correlations between X and Y . Therefore, $I(X, Y|Z)$ is non-zero. This implies that $I(X : Y : Z)$ is negative. This negative value captures a certain aspect of the correlations among the variables X, Y , and Z . We discuss an example for the quantum case later in this section.

In the case of three random variables, we can obtain many different classically equivalent expressions for $I(X : Y : Z)$. To achieve our goal, we obtain three different equivalent expressions. In one of the cases, the expression does not have any conditional entropy, while in the other two cases, the expression has conditional entropies with respect to one variable only or two variables only. In this way, after generalization to quantum domain, we can explore the quantum correlations of different partitions of the quantum system. Using the chain rule in the definition (4), we obtain

$$I(X : Y : Z) = H(X, Y) - H(Y|X) - H(X|Y) - H(X|Z) - H(Y|Z) + H(X, Y|Z). \quad (5)$$

We can convert the above expression that involves conditional entropies to that containing only entropies and joint entropies. We obtain

$$J(X : Y : Z) = [H(X) + H(Y) + H(Z)] - [H(X, Y) + H(X, Z) + H(Y, Z)] + H(X, Y, Z). \quad (6)$$

Using the chain rule $H(X, Y, Z) = H(Y, Z) + H(X|Y, Z)$, we can define the three-variable mutual information involving two-variable conditional entropies. This gives another equivalent expression

$$K(X : Y : Z) = [H(X) + H(Y) + H(Z)] - [H(X, Y) + H(X, Z)] + H(X|Y, Z). \quad (7)$$

These three expressions for the three-variable mutual information are classically equivalent, but not so in quantum domain. The difference of the three definitions can capture various aspects of the quantum correlations. In the next subsection, we will generalize these definitions to quantum domain.

In principle, for a given quantum system the amount of information one can extract from it depends on the nature and choice of the measurement. However, it has been shown [17] that the discord does not change if we use POVM measurements instead of projective measurements. As dissension can be written in terms of the discord, as shown below, one may expect that the situation will remain same. Therefore, in the next section we built up our definitions on the basis of projective measurement. But one can choose POVM measurements also.

Quantum Dissension for One-Particle Projective Measurement

Here, we extend the definitions of the three-variable mutual information $I(X : Y : Z)$ and $J(X : Y : Z)$ to the quantum domain. Let us consider a three-qubit state ρ_{XYZ} , where X, Y, Z refer to the first, second and the third qubit. The extension of the definition of $J(X : Y : Z)$ is straightforward. It is obtained by replacing the random variables by the density matrices and the Shannon entropies by the Von Neumann entropies. The extension of the expression for $I(X : Y : Z)$ requires appropriate extension of the conditional entropies. It is given by

$$I(X : Y : Z) = H(X, Y) - H(Y|\{\pi_j^X\}) - H(X|\{\pi_j^Y\}) - H(X|\{\pi_j^Z\}) - H(Y|\{\pi_j^Z\}) + H(X, Y|\{\pi_j^Z\}), \quad (8)$$

where $H(X|\{\pi_j^Y\}) = \sum_j p_j H(\rho_{X|\pi_j^Y})$, $\rho_{X|\pi_j^Y} = \frac{\pi_j^Y \rho_{XY} \pi_j^Y}{\text{Tr}(\pi_j^Y \rho_{XY})}$ and p_j is the probability of obtaining the j th outcome. Here, $H(X|\{\pi_j^Y\})$ is the average Von Neumann entropy of the qubit X , when the projective measurement is done on the subsystem Y in the general basis $\{|u_1\rangle = \cos(t)|0\rangle + \sin(t)|1\rangle, |u_2\rangle = \sin(t)|0\rangle - \cos(t)|1\rangle\}$ (where $t \in [0, 2\pi]$). Similarly, one can write down the equivalent expressions for $H(X|\{\pi_j^Z\})$, $H(Y|\{\pi_j^Z\})$, $H(Y|\{\pi_j^X\})$. Furthermore, $H(X, Y|\{\pi_j^Z\}) = \sum_j p_j H(\rho_{X,Y|\pi_j^Z})$, $\rho_{X,Y|\pi_j^Z} = \frac{\pi_j^Z \rho_{XYZ} \pi_j^Z}{\text{Tr}(\pi_j^Z \rho_{XYZ})}$. It is the average Von Neumann entropy of the subsystem 'XY', when the projective measurement is carried out on the qubit Z . $H(X, Y)$ refers to Von Neumann entropy of the density matrix $\rho_{X,Y}$. To define dissension for the single particle projective measurement, we consider the difference between $I(X : Y : Z)$ and $J(X : Y : Z)$. This difference is given by

$$\begin{aligned} D_1(X : Y : Z) &= I(X : Y : Z) - J(X : Y : Z) \\ &= H(X, Y|\{\pi_j^Z\}) + [H(X, Z) + H(Y, Z) \\ &\quad + 2H(X, Y)] - H(X, Y, Z) - [H(X|\{\pi_j^Y\}) \\ &\quad + H(X|\{\pi_j^Z\}) + H(Y|\{\pi_j^Z\}) + H(Y|\{\pi_j^X\})] \\ &\quad - [H(X) + H(Y) + H(Z)]. \end{aligned} \quad (9)$$

One can minimize this over all possible one-particle measurement projectors. So mathematically the expression for the dissension is given by, $\delta_1 = \min(D_1(X : Y : Z))$. For single-particle projective measurements, the above expression is the most general one in the sense that it includes all possible one-particle projective measurements. As a consequence of which the dissension δ_1 , may reveal the maximum possible quantum correlations. We note that dissension is not symmetric with respect to the permutations of the subsystems X, Y and Z , as in the case of discord.

Lemma 1: For an arbitrary pure three-qubit state $J(X : Y : Z) = 0$. Therefore, $D_1 = I(X : Y : Z)$.

Proof: For a pure three-qubit state $H(X, Y, Z) = 0$. This is because ρ_{XYZ} being a pure state has no uncertainty. Furthermore, Von Neumann entropies for the subsystems are related as $H(X) = H(Y, Z)$, $H(Y) = H(X, Z)$, and $H(Z) = H(X, Y)$. Therefore, $J(X : Y : Z) = 0$ and $D_1 = I(X : Y : Z)$ for a pure three-qubit state.

Lemma 2: For an arbitrary pure three-qubit system, $H(X, Y|Z) = H(X, Z|Y) = H(Y, Z|X) = 0$.

Proof: In the quantum domain, $H(X, Y|Z) = H(X, Y|\{\pi_j^Z\}) = \sum_j p_j H(\rho_{X,Y|\pi_j^Z})$. Here, $\rho_{X,Y|\pi_j^Z}$ is the density matrix of the system after the measurement has been performed on the subsystem Z . After the measurement, for each projector, the state of the subsystem XY is a pure state. Therefore, the Von Neumann entropy $H(X, Y|Z) = H(\rho_{X,Y|\pi_j^Z}) = 0$. Similarly, von Neumann entropies $H(X, Z|Y)$ and $H(Y, Z|X)$ are also zero.

Quantum Dissension for Two-Particle Projective Measurement

For a three-qubit system, we can also make measurement on a two-qubit subsystem. This will probe different aspects of the quantum correlations. We also define dissension involving all two-qubit measurements. For this purpose, we define the quantum analogue of the classical mutual information $K(X : Y : Z)$ as follows.

$$K(X : Y : Z) = [H(X) + H(Y) + H(Z)] - [H(X, Y) + H(X, Z)] + H(X|\{\pi_j^{Y,Z}\}) \quad (10)$$

where $H(X|\{\pi_j^{Y,Z}\}) = \sum_j p_j H(\rho_{X|\pi_j^{Y,Z}})$; $\rho_{X|\pi_j^{Y,Z}} = \frac{\pi_j^{Y,Z} \rho_{XYZ} \pi_j^{Y,Z}}{\text{Tr}(\pi_j^{Y,Z} \rho_{XYZ})}$. It is the average Von Neumann entropy of the qubit X , when the projective measurement is carried out on the subsystem 'YZ' in the general basis $\{|v_1\rangle = \cos(t)|00\rangle + \sin(t)|11\rangle, |v_2\rangle = -\sin(t)|00\rangle + \cos(t)|11\rangle, |v_3\rangle = \cos(t)|01\rangle + \sin(t)|10\rangle, |v_4\rangle = -\sin(t)|01\rangle + \cos(t)|10\rangle\}$ and p_j is the probability of obtaining the j th outcome. Here, $H(X), H(Y), H(Z), H(X, Y), H(X, Z)$ represents the Von Neumann entropies of the subsystems, $\rho_X, \rho_Y, \rho_Z, \rho_{XY}, \rho_{XZ}$, respectively. To define the dissension, we take the difference

$$D_2(X : Y : Z) = K(X : Y : Z) - J(X : Y : Z) = H(X|\{\pi_j^{Y,Z}\}) + H(Y, Z) - H(X, Y, Z). \quad (11)$$

Like one-particle projective measurement case, here also we define dissension as, $\delta_2 = \min(D_2(X : Y : Z))$. Furthermore, as in the case of discord, this quantity is not symmetric under the permutations of X, Y and Z . If we choose to make a measurement on 'XY' subsystem, instead of 'YZ' subsystem, we can obtain the corresponding expression for

D_2 by interchanging X and Z . Similarly, we can interchange X and Y if we make a measurement on 'XZ' subsystem. In each case, D_2 will reduce to discord with appropriate bipartite split, as discussed below in Lemma 4.

Lemma 3: For an arbitrary pure three-qubit system, $H(X|Y, Z) = H(Y|X, Z) = H(Z|X, Y) = 0$. Therefore, $D_2 = H(X)$ and the dissension is given by the Von Neumann entropy of the bipartite partition.

Proof: We have defined: $H(X|Y, Z) = H(X|\{\pi_j^{YZ}\}) = \sum_j p_j H(\rho_{X|\pi_j^{YZ}})$. After the measurement, the system is in a product state of the state of the X and the projected state of the YZ subsystem, which is a pure state. Therefore, its Von Neumann entropy is zero, i.e., $H(\rho_{X|\pi_j^{YZ}}) = 0$ for all j . It implies that $H(X|Y, Z) = 0$. Similarly, $H(Y|X, Z) = H(Z|X, Y) = 0$. Furthermore, for any pure three-qubit state, $H(X) = H(Y, Z)$ and $H(X, Y, Z) = 0$. Therefore, $D_2 = H(X)$.

We note that the above Lemma is for three-qubit case. For multi-qubit case one can define the dissension along the same line. In that case the expression will be non-trivial and it is not going to be the Von Neumann entropy of the bipartite partition [19].

Lemma 4: Dissension is related to discord as :

$$\begin{aligned} D_1(X : Y : Z) &= D(X, Y : Z) - D(X : Z) - D(Y : Z) \\ &\quad - D(X : Y) - D(Y : X), \quad (12) \\ D_2(X : Y : Z) &= D(X : Y, Z). \quad (13) \end{aligned}$$

Therefore, in the case of a tripartite state, D_2 is just discord with a bipartite split of the system.

Proof: The proof of this lemma is straightforward. To prove the relation (12), we start with the definition of D_1 which is given in (9). We now group terms together and rewrite them in terms of discord D , using (2). We notice that the fourth, sixth, ninth, tenth and eleventh terms in (9) combine to give the last two terms of (12). By adding and subtracting $H(Z)$, we can get the remaining three terms. The relation for D_2 is obvious from (2) and (11).

Quantum Mutual Information Can be Negative

Classical three-variable mutual information is defined in Equations (3) and (4). Its generalization to quantum domain has been discussed in this section. As discussed earlier, when $I(X : Y)$ is smaller than $I(X, Y|Z)$, then $I(X : Y : Z)$ can be negative. It happens when knowing the state Z enhances the quantum correlations between states X and Y . For a simple example, consider pure three-qubit GHZ state. If we trace out one qubit (say Z), the reduced density matrix is a mixture of product states. If we compute mutual information with measurement in Hadamard basis, then the mutual information

$I(X : Y)$ is zero. However, a measurement in Hadamard basis on the qubit Z reduces the subsystem of the qubits X and Y to a Bell State. One can compute that $I(X, Y|Z) = 2$. As a result $I(X : Y : Z) = -2$. Here, we see that a measurement in appropriate basis on a subsystem can enhance the correlation in the rest of the system. So the conditional mutual information can be larger than the mutual information, thus making three-variable mutual information negative. This should not be regarded as a drawback of the definition. On the contrary this may give some new insight into the true nature of quantum correlations in multipartite setting. We may also note that different measures of correlations for multipartite situations may capture different aspects of the quantumness.

Comparison with other measures for three-qubit states

In reference [13], the authors gave an unified view of total correlation for multi party states in terms of relative entropy. They defined entanglement (E), discord (D) and dissonance (Q) for a given state ρ as $E = \min_{\sigma \in \mathcal{S}} S(\rho||\sigma)$, $D = \min_{\xi \in \mathcal{C}} S(\rho||\xi)$, $Q = \min_{\xi \in \mathcal{C}} S(\sigma||\xi)$. Here $S(\cdot||\cdot)$ represents the relative entropy. The states σ and ξ are the closest states to the the state ρ in the sets of separable states (\mathcal{S}) and classical states (\mathcal{C}) respectively. However, our approach is based on three-variable mutual information, not on relative entropy. We are looking at the information contained in a specific state, while in the case of relative entropy, the distances from separable or classical states are used. Our philosophy is that one number is not enough to characterize the quantum correlations of a multipartite state. Therefore, we consider two separate types of measurements. Depending on what information processing task one wishes to carry out, one set of measures may be more useful than the others.

3. QUANTUM DISSENSATION FOR PURE THREE-QUBIT STATES

In this section we present quantum dissension for pure three-qubit states. As mentioned earlier, one can carry out both the single-particle and two-particle projective measurements in the most general basis. We illustrate the usefulness of these definitions by considering pure three-qubit GHZ and W states.

Quantum Dissension for GHZ state

Let us consider a pure three-qubit GHZ state

$$|GHZ\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \quad (14)$$

First, we calculate the dissension when the projective measurement is carried out on one particle. After tracing out two

qubits, the one-qubit density matrices representing the individual subsystems are given by $\rho_A = \rho_B = \rho_C = \frac{1}{2}$ with the von Neumann entropies equal to one, i.e., $H(A) = H(B) = H(C) = 1$. Similarly, by tracing out any one of the three qubits, we obtain the reduced density matrices of the subsystems as

$$\rho_{AB} = \rho_{BC} = \rho_{CA} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|). \quad (15)$$

Consequently, the entropies of these density matrices are given by $H(AB) = H(BC) = H(CA) = 1$. Since $|GHZ\rangle$ is a pure state, so the joint Von Neumann entropy $H(ABC)$ vanishes.

Next, we consider the conditional entropy of the two systems, when the projective measurement is done on the third system and we find it to be zero, i.e., $H(AB|\{\pi_j^C\}) = 0$ (using Lemma 2). We also find the conditional entropies of one qubit system when the projective measurement is done on any one of the other two-qubit systems. These are given by

$$\begin{aligned} H(A|\{\pi_j^B\}) &= H(A|\{\pi_j^C\}) = \\ H(B|\{\pi_j^C\}) &= H(B|\{\pi_j^A\}) = \\ &= -\left(\frac{1 - \cos(2t)}{2}\right) \log_2 \frac{1 - \cos(2t)}{2} \\ &\quad - \left(\frac{1 + \cos(2t)}{2}\right) \log_2 \frac{1 + \cos(2t)}{2}. \end{aligned} \quad (16)$$

Consequently, the expression for D_1 in case of GHZ state is given by

$$\begin{aligned} D_1 &= 1 + 4 \left[\frac{1 - \cos(2t)}{2} \log_2 \frac{1 - \cos(2t)}{2} \right. \\ &\quad \left. + \frac{1 + \cos(2t)}{2} \log_2 \frac{1 + \cos(2t)}{2} \right]. \end{aligned} \quad (17)$$

We have plotted D_1 for the GHZ state for the single-particle projective measurement as a function of t in the Figure 1 (i). It is a oscillating function which varies between $[-3, 1]$. By minimizing over all possible bases, the dissension is given by $\delta_1 = -3$. The part of this calculation was performed using the mathematica package QDENSITY [18]. For GHZ state, in case of two-particle projective measurement, the conditional entropy is zero and the dissension reduces to the bipartite entanglement present in the system and is equal to one (see Lemma 3). Therefore dissension for the state is $(\delta_1, \delta_2) = (-3.00, 1.00)$.

Quantum Dissension for W state

Here, we consider another inequivalent class of pure three-qubit state which is known as W state. This is given by

$$|W\rangle_{ABC} = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle). \quad (18)$$

The density matrices representing each of the one qubit subsystem of the three qubit W state are given by

$$\rho_A = \rho_B = \rho_C = \frac{1}{3}[2|0\rangle\langle 0| + |1\rangle\langle 1|]. \quad (19)$$

The Von Neumann entropies of these density matrices are found to be $H(A) = H(B) = H(C) = 0.92$. Similarly, by tracing out any one qubit we obtain the two qubit density matrices representing two-qubit subsystem of the W state. These are given by

$$\rho_{AB} = \rho_{BC} = \rho_{CA} = \frac{1}{3} [|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|]. \quad (20)$$

Hence, the Von neumann entropies of the two-qubit subsystems are given by $H(AB) = H(BC) = H(AC) = 0.92$. Since the W state is a pure state, the joint Von Neumann entropy for the W state is zero, i.e., $H(ABC) = 0$. Then, we find out all other conditional entropic quantities required for finding out the dissension when the projective measurement is carried out on a single qubit system. The conditional entropy of any two-qubit system when the projective measurement is carried out on the third qubit is zero, i.e., $H(AB|\{\pi_j^C\}) = 0$ (see Lemma 2). In addition, the conditional entropies of one qubit systems when the projective measurement is done on any one of the remaining qubits, are given by

$$\begin{aligned} H(A|\{\pi_j^B\}) &= H(A|\{\pi_j^C\}) = H(B|\{\pi_j^C\}) = \\ H(B|\{\pi_j^A\}) &= 1 + \frac{1}{2} (p H(1+x_+, 1-x_+) \\ &\quad + (1-p) H(1+x_-, 1-x_-)), \end{aligned} \quad (21)$$

where $p = \frac{(3+\cos(2t))}{6}$ and,

$$H(x, y) = -x \log_2(x) - y \log_2(y), \quad (22)$$

$$x_{\pm} = \frac{\sqrt{(5 \pm 3 \cos(2t))(1 \mp \cos(2t))}}{(3 \pm \cos(2t))}. \quad (23)$$

On using the above equations, the expression for D_1 is given by

$$D_1 = H(AB) - 4H(A|\{\pi_j^B\}). \quad (24)$$

On substituting the values of these entropic quantities, the expression D_1 for the W state is plotted in Figure 1(ii). In this case, we see from the plot that the dissension is -1.74 . The computations here and below were performed using the mathematica package QDENSITY [18].

Like the GHZ state, for the two-particle projective measurements, the conditional entropy is zero and the dissension reduces to the bipartite entanglement present in the system. For the W state, it is equal to .92 (see Lemma 3). So the dissension for the W state is $(\delta_1, \delta_2) = (-1.74, 0.92)$.

4. QUANTUM DISSENSATION FOR MIXED THREE-QUBIT STATES

In this section, we analyze quantum dissension for three-qubit mixed states. We obtain D_1 and D_2 as a function of the

angle t and the classical probability of mixing a after carrying out the projective measurement in the most general basis. The case for the mixed state is illustrated by the pseudo-pure GHZ state with $\rho_{\text{GHZ}} = (1-a)\frac{I}{8} + a|GHZ\rangle\langle GHZ|$ and the pseudo-pure W state with $\rho_W = (1-a)\frac{I}{8} + a|W\rangle\langle W|$. (Here I is the random mixture.) These mixed states can be viewed as the generalization of the Werner state for the case of three-qubit states. Specifically, the mixed GHZ state can be thought of as a mixture of a random state and a "maximally" entangled state. As we shall see, the dissension for this state is non zero unless it is a random mixture.

Quantum Dissension for mixed GHZ state

Let us consider a three-qubit mixed GHZ state

$$\rho_{\text{GHZ}} = (1-a)\frac{I}{8} + a|GHZ\rangle\langle GHZ|. \quad (25)$$

After tracing out any two qubits, the reduced density matrices representing the individual subsystems are given by the random mixture, i.e., $\rho_A = \rho_B = \rho_C = \frac{I}{2}$. The Von Neumann entropies of these subsystems are all equal to one. Similarly, by tracing out a single qubit one can obtain the two-qubit density matrices given by

$$\begin{aligned} \rho_{AB} = \rho_{BC} = \rho_{CA} &= \frac{1+a}{4} [|00\rangle\langle 00| + |11\rangle\langle 11|] \\ &\quad + \frac{1-a}{4} [|01\rangle\langle 01| + |10\rangle\langle 10|] \end{aligned} \quad (26)$$

We have calculated D_1 and D_2 in general basis. Since the expression for the quantum dissensions for ρ_{GHZ} are quite involved and lengthy, we do not provide these analytic expressions in this manuscript. We have plotted the D_1 and D_2 in Figures 2(i), (iii). Here, we can see that that for $a = 1$, this gives us the back the D_1 of the GHZ state which is given in Figure 1(i). Furthermore the dissensions δ_1 and δ_2 are non zero for any non-zero values of a . This is like the two-qubit Werner state. We also notice that D_2 reduces to that of the pure GHZ state for $a = 1$ as expected.

Quantum Dissension for mixed W state

Here, we evaluate the quantum dissension for the mixed three-qubit W state. The state is given by

$$\rho_W = (1-a)\frac{I}{8} + a|W\rangle\langle W|. \quad (27)$$

After tracing out any two qubits, the reduced density matrices representing the one qubit subsystems are given by

$$\rho_A = \rho_B = \rho_C = \frac{3+a}{6} |0\rangle\langle 0| + \frac{3-a}{6} |1\rangle\langle 1|. \quad (28)$$

Similarly, one can trace out a single qubit to obtain the density matrices of two-qubit subsystems as

$$\begin{aligned}\rho_{AB} &= \rho_{BC} = \rho_{CA} \\ &= \left[\frac{3+a}{12}\right][|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|] \\ &+ \left[\frac{1-a}{4}\right][|11\rangle\langle 11| + \frac{a}{3}[|01\rangle\langle 10| + |10\rangle\langle 01|]].\end{aligned}\quad (29)$$

We evaluate the Von Neumann entropies of these above density matrices, all the conditional entropies (for both the one-particle and the two-particle measurement) and the joint entropy of three-qubit Mixed W state. Here also, we do not provide analytic expressions for dissension for ρ_W as they are quite long and beyond the scope of the present paper. In the Figures 2(ii) and (iv), we show D_1 and D_2 as a function of the classical probability of mixing a as well as the angle t . It is evident from the Figure 2 (ii) that for $a = 1$, it gives us back the curve which is obtained in the Figure 1 (ii). Similarly, in the Figure 2(iv), D_2 reduces to that of the pure W state for $a = 1$.

Quantum Dissension for a Biseparable State

In this subsection, we give an example of a state for which δ_1 is non-zero, but δ_2 is zero. The state is

$$\begin{aligned}\rho_{bi} &= a(|000\rangle\langle 000| + |011\rangle\langle 011| + |000\rangle\langle 011| + \\ &|011\rangle\langle 000|) + b(|001\rangle\langle 001| + |010\rangle\langle 010| - \\ &|001\rangle\langle 010| - |010\rangle\langle 001|),\end{aligned}\quad (30)$$

where $a+b = \frac{1}{2}$. One can compute the dissension for this state as described above. One would find $(\delta_1, \delta_2) = (1.00, 0.00)$. One can understand these values by rewriting this state as a biseparable state

$$\rho_{bi} = 2a|0\rangle|\varphi^+\rangle\langle 0|\langle\varphi^+| + 2b|0\rangle|\psi^-\rangle\langle 0|\langle\psi^-|, \quad (31)$$

where $|\varphi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. For this state, D_2 is zero since the state is a product state in the partition 1 : (23). However, D_1 is due to single-particle measurement and it is non-zero. One can compute the concurrence for the qubit subsystem 2 and 3 and find that this subsystem is indeed entangled. Therefore, as argued earlier, the correlations of a tripartite state needs to be characterized by both (δ_1, δ_2) . This example also shows that one needs to compute D_2 for all bipartite partitions.

5. CONCLUSION

In this paper we have introduced the notion of Quantum Dissension for multipartite systems. Starting from three-

variable mutual information definition, we obtained three equivalent expressions. When these expressions are generalized to quantum domain, then from the difference between two such expressions, we obtain two kinds of dissensions. Classically, these differences are zero but in quantum domain they are not. This generalizes the notion of quantum discord to tripartite systems. We have multiple definitions of the dissension that can capture different aspects of the quantum correlations. We have illustrated the notion of quantum dissension using three-qubit pure and mixed states. Specifically, we have taken pure and mixed GHZ as well as W states and calculated the dissensions. We have focused on two such definitions of dissensions. One of the definitions δ_1 involves only one-particle measurements, while the second definition δ_2 involves only two-particle measurements. We find that dissension can be negative. It just reflects the fact that a measurement on a subsystem can enhance the correlation of the rest of the system. This is a new feature that emerges when we deal with multiparticle systems. One can interpret the dissension as the information contained in the multiparticle quantum state that cannot be extracted by one-particle or two-particle measurements. We have also obtained the relations between D_1 , D_2 and the discord D . It turns out that D_2 is just the discord for the bipartite split. However, if we generalize discord to a system of more than three qubits, then appropriate D_2 would not be discord (and even can be negative), similar to the quantity like D_1 [19]. We also note that the dissension is non-zero for all values of the classical mixing parameter for the mixed GHZ and W states. The mixed GHZ state that we have considered can be thought of as a generalization of the two-qubit Werner state. In the case of Werner state, entanglement vanishes for certain values of the mixing parameter, but discord is non-zero. We have also discussed one example where the dissension due to one-particle measurement is non-zero but that due to two-particle measurement is zero. In future, one may like to put bounds on the values of the dissension for three qubit system and higher dimensional systems. One may also like to relate the notion of dissension to the successes of various communication protocols. This may help in providing an operational meaning to dissension. Following our approach, one can generalize the notion of dissension beyond three-qubit system [19]. It will be interesting to see if the multipartite dissensions play any role in giving extra power to quantum computing in the absence of the entanglement.

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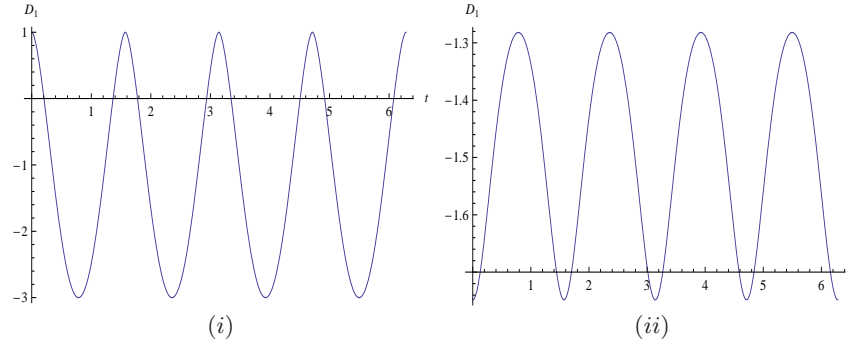


FIG. 1: D_1 for (i) three-qubit pure GHZ state and (ii) three-qubit pure W state .

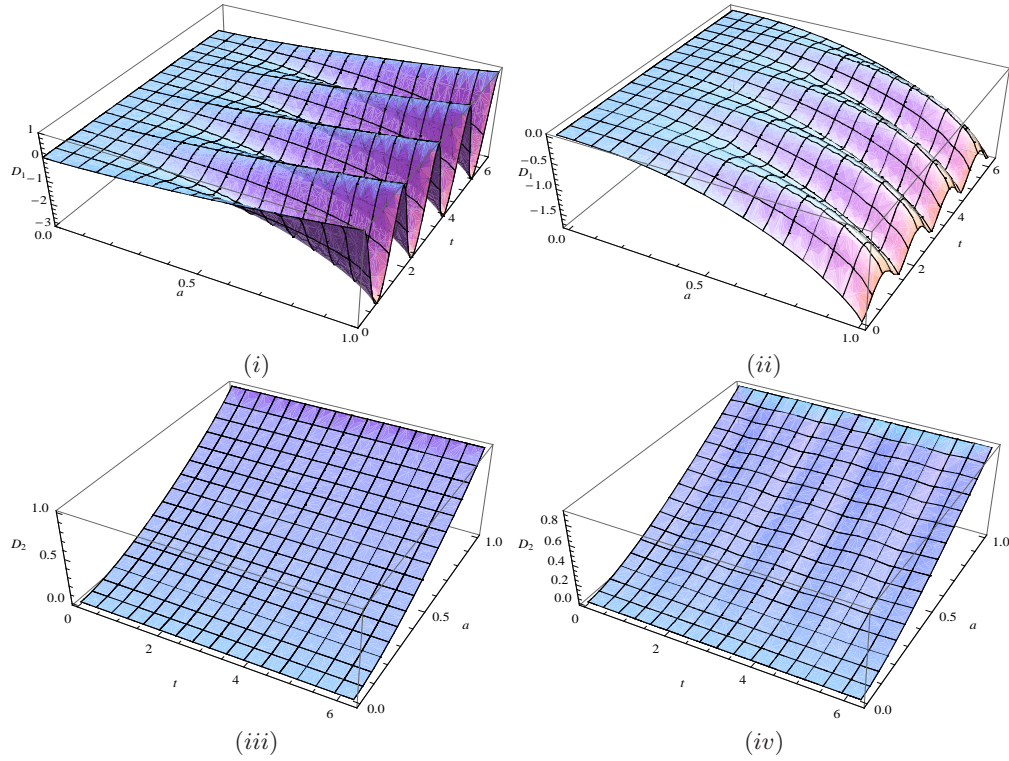


FIG. 2: D_1 and D_2 for three-qubit Mixed GHZ state (figures (i), (iii)) and three-qubit Mixed W state (figures (ii), (iv)).

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